

Robust rational approximation with barycentric representations

Silviu Filip, CAIRN team, Univ Rennes, Inria, CNRS, IRISA
joint work with Bernhard Beckermann, Yuji Nakatsukasa
and Lloyd N. Trefethen

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Rational functions

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→ powerful **approximations** near singularities or on unbounded domains

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Some applications:

- elementary + special functions
- recursive filter design
- matrix exponentials & stiff PDEs
- optimal control problems
- ...

Rational minimax approximation

Input: $f \in \mathcal{C}([a, b])$, target type $(m, n) \in \mathbb{N}^2$

Output: $r^* \in \mathcal{R}_{m,n} = \left\{ \frac{p}{q}, p \in \mathbb{R}_m[x], q \in \mathbb{R}_n[x] \right\}$ s.t.

$\|f - r^*\|_\infty$ is **minimal**.

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- **Alternation Theorem** [Achieser 1930]:

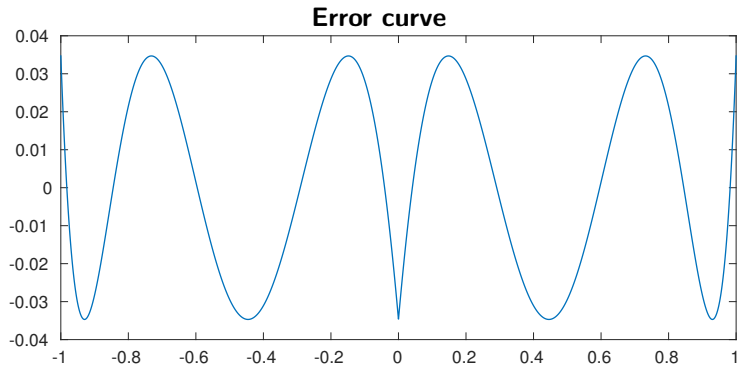
→ $f - r^*$ equioscillates at least $m + n + 2 - d$ times

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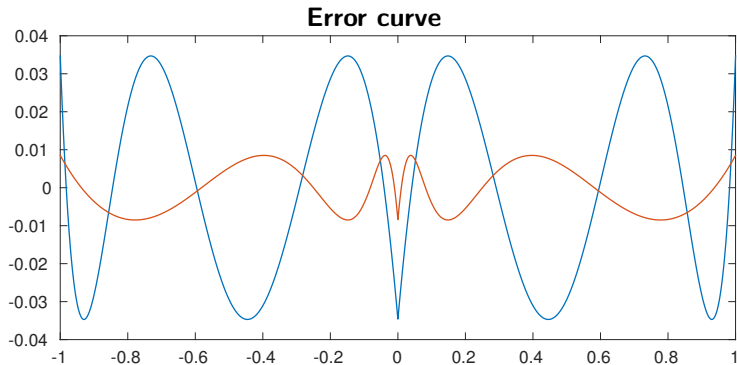
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- type (4, 4) rational function



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→ asymptotic behavior

$$\begin{aligned} E_{n,0}(f) &\sim \beta/n, & \beta &= 0.2801\dots & [\text{Varga \& Carpenter 1985}] \\ E_{n,n}(f) &\sim 8e^{-\sqrt{n}}, & & & [\text{Newman 1964, Stahl 1993}] \end{aligned}$$

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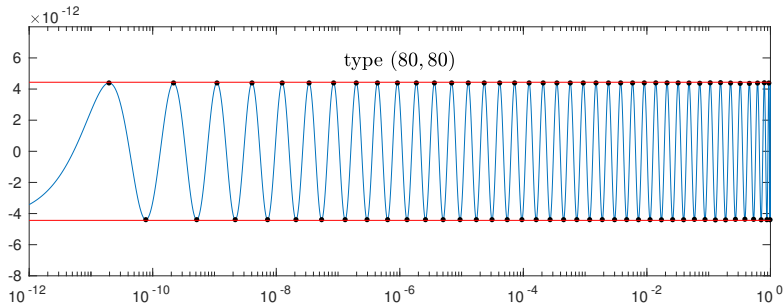
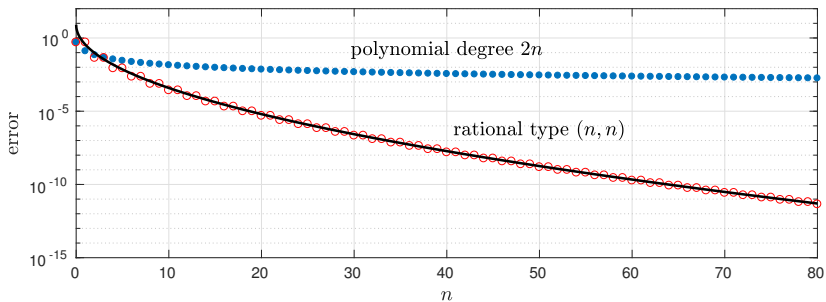
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- Chebfun (Matlab): `minimax`

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Why use adaptive barycentric formulas?

→ problem dependent $\{t_k\} \Rightarrow$ well conditioned representation

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Step 2: find $r \in \mathcal{R}_{n,n}$ and $\lambda \in \mathbb{R}$ s.t.

$$f(x_k) - r(x_k) = (-1)^{k+1} \lambda, \quad k = 0, \dots, 2n + 1$$

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Step 3: among local extrema of $f - r$, take $2n + 2$ new points

$$a \leq x'_0 < \dots < x'_{2n+1} \leq b,$$

$f - r$ alternates in sign + at least one global extrema over $[a, b]$ and

$$|f(x'_k) - r(x'_k)| \geq |\lambda|, \quad k = 0, \dots, 2n + 1$$

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Convergence:

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What can go wrong?

- no pole-free solution in **Step 2**

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Our strategy:

- use Carathéodory-Fejér (CF) approximation [Trefethen & Gutknecht 1983]
- AAA-Lawson approx. (adaptively re-weighted least squares AAA variant)
- extrapolation from lower degree approx. $((2, 2), (3, 3), (4, 4), \dots)$

Step 2: find r

→ find $r = N/D \in \mathcal{R}_{n,n}$ s.t.

$$N(x_k) = D(x_k)(f(x_k) - (-1)^{k+1}\lambda), \quad k = 0, \dots, 2n + 1$$

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→ matrix form

$$C\alpha = \left(\begin{array}{c} \left[\begin{array}{cccc} f(x_0) & & & \\ & f(x_1) & & \\ & & \ddots & \\ & & & f(x_{2n+1}) \end{array} \right] - \lambda \left[\begin{array}{cccc} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & \ddots \end{array} \right] \end{array} \right) C\beta,$$

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→ generalized eigenvalue problem

$$\begin{bmatrix} C & -FC \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \lambda \begin{bmatrix} 0 & -SC \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$F = \text{diag}(f(x_k)), S = \text{diag}((-1)^{k+1})$

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$$Q_1^T (SF) Q_1 R \beta = \lambda R \beta,$$

where $\omega_x(x) = \prod_{k=0}^{2n+1} (x - x_k)$, $\omega_t(x) = \prod_{j=0}^n (x - t_j)$,

$$\Delta = \text{diag} \left(\frac{\omega_t(x_0)^2}{\omega'_x(x_0)}, \dots, \frac{\omega_t(x_{2n+1})^2}{\omega'_x(x_{2n+1})} \right)$$

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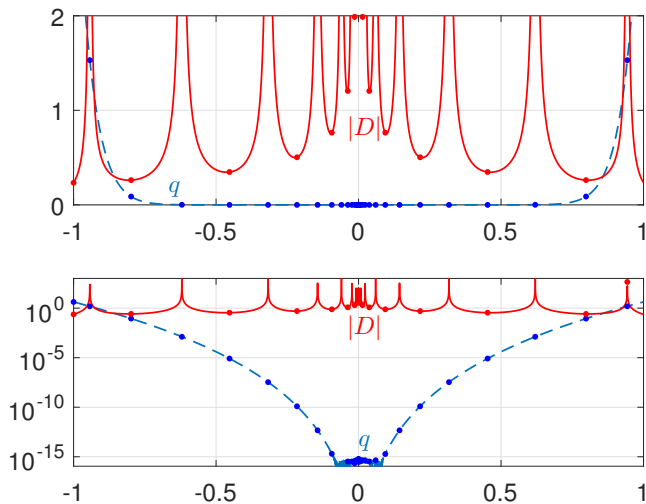
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→ we show that this problem is usually well conditioned if

$$t_k = x_{2k+1}, \quad k = 0, \dots, n$$

Example: $f(x) = |x|$, $x \in [-1, 1]$, type (20, 20)

monomial p/q vs barycentric N/D



→ dots represent the x_k 's

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→ Chebyshev interpolants of $e(x) = f(x) - r(x)$ on each subinterval

→ colleague matrix root finding [Specht, Good]

DEMO

Conclusion

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→ the details:

B. Beckermann, S.-I. Filip, Y. Nakatsukasa, L. N. Trefethen, *Rational minimax approximation via adaptive barycentric representations*, arXiv:1705.10132

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